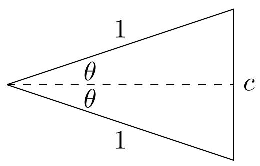
2301. A quadrilateral $PQRS$ has the property that the diagonals PR and QS each divide the quadrilateral into two regions of equal area. Prove that $PQRS$ is a parallelogram.

2302. Solve the equation $\dfrac{1}{2^{2x+1}} = 4^{6x}$.

2303. Use the triangle below to prove the double-angle formula $\cos 2\theta \equiv 1 - 2\sin^2\theta$.



2304. A hand of five cards is dealt from a standard deck. Find, to 3sf, the probability that the hand consists of the numbers $3, 4, 5, 6, 7$, of any suit.

2305. A set of (x_i, y_i, z_i) data points has been collected. One hypothesis test is conducted for correlation between x and y, and another hypothesis test is conducted for correlation between x and z. Both show significant evidence of correlation. Explain whether or not this guarantees that a hypothesis test between x and z will yield a significant result.

2306. Prove that, if two tangent lines are drawn to the parabola $y = x^2 + 2x$ at $x = a$ and $x = b$, then they meet at $x = \frac{1}{2}(a + b)$.

2307. An ellipse is given, in parametric vector form, by $\mathbf{r} = a\cos t\,\mathbf{i} + b\sin t\,\mathbf{j}$, for $t \in [0, 360°)$. Without using calculus, determine a formula for the area of the ellipse.

2308. A value of x is randomly chosen on the interval $[0, 1]$. Find the probability that $|3x - 1| \leq 1$.

2309. State whether the following functions have a sign change at $x = 0$. You don't need to consider if the functions are defined at $x = 0$ itself.

(a) $\mathrm{h}(x) = x\sin x$,
(b) $\mathrm{h}(x) = x^3\sin^2 x$,
(c) $\mathrm{h}(x) = x\sec x$,
(d) $\mathrm{h}(x) = (x + 1)\operatorname{cosec} x$.

2310. Write the following as a single simplified fraction:

$$\frac{1}{x^2 + 2x + 1} + \frac{1}{x} - \frac{1}{x + 1}.$$

2311. Find $\displaystyle\int \operatorname{cosec}^2 x\,dx$.

2312. In a game at a fete, customers roll a fair six-sided die. If the die comes up six, the customer wins; if it comes up four or five, they roll again; otherwise, they lose.

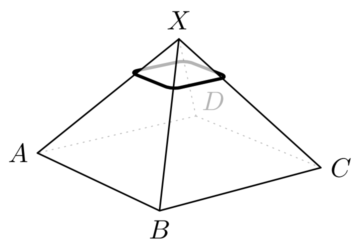
(a) Draw a tree diagram representing the first and potential second rolls.
(b) Find the probability that the customer wins within the first two rolls.
(c) Show that $\mathbb{P}(\text{win}) = \displaystyle\sum_{i=1}^{\infty} \tfrac{1}{6} \times \tfrac{1}{3}^{i-1}$.
(d) Hence, show that $\mathbb{P}(\text{win}) = \frac{1}{4}$.
(e) Explain how your result in (d) could have been reached immediately as a probability of the form $p = \frac{\text{successful}}{\text{total}}$ outcomes.

2313. Explain why the regions of the (x, y) plane defined by $x^2 + y^2 < 1$ and $x^2 + y^2 \leq 1$ have the same area.

2314. Write the following in the form $k\log_4 x$:

(a) $\log_2 x$,
(b) $\log_{16} x$,
(c) $\log_8 x$.

2315. The square-based pyramid shown below is formed of eight edges of unit length. A loop of string of unit length has been placed, taut and horizontal, around the pyramid.



Determine the vertical height of the loop of string above the base $ABCD$.

2316. Sketch a linear graph $x = \mathrm{h}(y)$ for which

$$\int_2^3 \mathrm{h}(y)\,dy = -6, \quad \mathrm{h}(0) = 0.$$

2317. In this question, use $g = 10$. Projectiles are fired at speed 20 ms^{-1} from an origin, at angle $\arcsin {}^3\!/_5$ above the positive x axis, one after another. The time delay between each is 1 second.

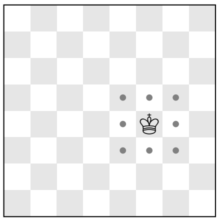
(a) Find the coordinates of the highest point reached by the projectiles.
(b) Determine the shortest distance between two successive projectiles during their motion.

2318. Linear functions are given as $\mathrm{f}(x) = ax + b$ and $\mathrm{g}(x) = cx + d$. Show that, if $\mathrm{fg}(x) \equiv \mathrm{gf}(x)$, then

$$\frac{a - 1}{c - 1} = \frac{b}{d}.$$

2319. A *Heronian triangle* is one with integer sides and area. Show that $(3, 25, 26)$ is a Heronian triangle.

2320. In chess, a king threatens squares as shown. Find the probability that, if two kings are placed on a chessboard at random, they threaten each other.



2321. State, with a reason, whether $y = k$ intersects the following curves:

(a) $y = \operatorname{cosec} x + k$,

(b) $y = \sec x + k$,

(c) $y = \cot x + k$.

2322. Three forces **P**, **Q**, **R** act on an object, which is in equilibrium. The magnitudes of the forces are 6, 10 and 11 Newtons respectively. Find the angle between forces **P** and **Q**.

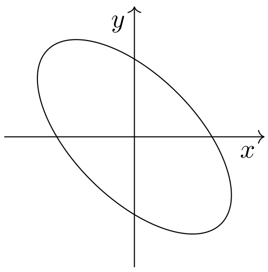
2323. Functions f and g are invertible with domain and codomain $\mathbb{R}$. Find expressions, using f^{-1} and g^{-1}, for the inverses of the following:

(a) $\text{f}(2x - 1)$,

(b) $\text{fg}(x)$,

(c) $\text{g}^{-1}\text{f}(x)$.

2324. The equation $x^2 + xy + y^2 = 1$ defines an ellipse.



(a) Find $\dfrac{dy}{dx}$ in terms of x and y.

(b) Show that the tangents to the ellipse at its x intercepts cross the y axis at ± 2.

2325. Two dice are rolled. State, with a reason, which, if either, of the following has greater probability:

① the difference is five,

② the sum is two.

2326. An AP begins $a, b, 2, ...$ and a GP begins $b, a, 9, ...$ Find all possible values of a and b.

2327. By finding the area under a velocity-time graph, prove the formula $s = ut + \frac{1}{2}at^2$.

2328. In this question, the unit of angle is the radian. Show that $(\sin^{-1} x)^2 + \sin^{-1} x = 12$ has no roots.

2329. State, with a reason, whether the following holds: "In a binomial hypothesis test, if $\mathbb{P}(x \le k) < \frac{1}{20}$, then, at the 5% level, k lies in the critical region for the test."

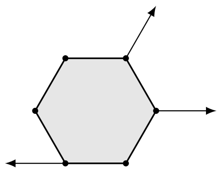
2330. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ (if any) links statements ① and ② concerning a polynomial function f:

① $\text{f}(x)$ does not have a factor of $(x - a)$,

② $\text{f}(a) \neq 0$.

2331. The function $\text{f} : x \mapsto x^8 + x^7$ has second derivative zero at $x = 0$. Determine whether or not this is a point of inflection.

2332. A rigid hexagonal prism stands vertically. Three horizontal forces are exerted on it, acting parallel to sides of the hexagon, as shown in plan view:



Show that, whatever the (non-zero) magnitudes of these forces, the prism must both translate and rotate.

2333. Show that the area of the region enclosed by the curves $y = x^2 + kx$ and $y = x^3 + kx$ is independent of the constant k.

2334. You are given that

$$\sin \tfrac{5\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

(a) Use the identity $1 + \cot^2 \theta \equiv \operatorname{cosec}^2 \theta$ to prove the following result:

$$\tan \tfrac{5\pi}{12} = \frac{1}{\sqrt{7 - 4\sqrt{3}}},$$

(b) Hence, verify that $\tan \frac{5\pi}{12} = 2 + \sqrt{3}$.

2335. Four variables are linked by $c = a + b$, $d = a - b$. Express a and b in terms of c and d.

2336. A student writes: "Friction acts to oppose motion or potential motion. When a runner does laps of a circular track, acceleration is towards the centre, so friction must be acting outwards, away from the centre." Criticise this argument.

2337. A function f is said to be "increasing everywhere" if, for every x in the domain of f, $f'(x) > 0$. Let f be such a function. Either prove or disprove the following statements:

(a) The graph $y = f(x)$ has no stationary points.

(b) For $a, b \in \mathbb{R}$, $a > b \implies f(a) > f(b)$.

(c) $f(x) = 0$ has a maximum of one root.

2338. Show that it is impossible to find constants A, B, C such that the following is an identity:

$$\frac{2x-1}{x^4+x^2} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^2+1}.$$

2339. The product rule for differentiation states that

$$(uv)' = u'v + uv'.$$

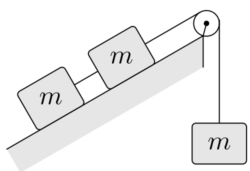
Assuming this rule, prove that

$$(uvw)' = u'vw + uv'w + uvw'.$$

2340. From a committee of sixteen people, a chairperson and two secretaries are chosen. Find the number of different ways in which this can be done.

2341. "The curves $x^2+y^2 = k$ and $(x-k)^2+(y-k)^2 = k$ are tangent for exactly one $k \in (0, \infty)$." True or false?

2342. Three masses are connected by light, inextensible strings, one of which is passed over a smooth, light, fixed pulley as shown in the diagram. The slope is smooth, at inclination 30°, and the masses are released from rest.



(a) Find the tension in the shorter string.

(b) Explain whether your answer would have been different had

i. the pulley not been smooth,

ii. the slope not been smooth.

2343. A sequence is given iteratively by

$$A_{n+1} = \tfrac{2}{3}A_n, \quad A_1 = 1000.$$

Find the value of k in the first pair of consecutive terms A_k, A_{k+1} for which $A_k - A_{k+1} < 1$.

2344. Find all possible values of $\sin x$ and $\cos y$, given that the following equations hold:

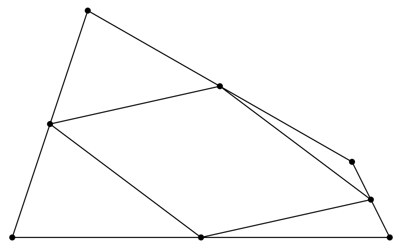
$$\begin{aligned} 0 &= 2\sin(x-y) + 2\sin(x+y) + 1, \\ 0 &= \sin x + \cos y. \end{aligned}$$

2345. For a circle $x^2 + y^2 = r^2$, differentiate implicitly to show that $\frac{dy}{dx} = -x/y$. Interpret this result in terms of the gradient of the radius.

2346. Solve the following equation, giving your answer in exact form:

$$\frac{1}{2^x+1} + \frac{1}{2^x-1} = \frac{1}{4}.$$

2347. The midpoints of the sides of quadrilateral $ABCD$ form a new quadrilateral $PQRS$.



By using vectors, or otherwise, prove that $PQRS$ is a parallelogram.

2348. Region R is defined by those points simultaneously satisfying the inequalities

$$\begin{aligned} (x-4)^2 + (y-7)^2 &< 30, \\ (x-3)^2 + (y-5)^2 &> 6. \end{aligned}$$

(a) Show that the boundary equations of these two inequalities do not intersect.

(b) Hence, show that R has area 24π.

2349. Show that, at any roots of a polynomial function f, the graphs $y = f(x)$ and $y = \ln\big(f(x)+1\big)$ are tangent.

2350. Find simplified expressions, in terms of constants a and b, for the coordinates of the vertex of the graph $y = (x-a)(x-b)$.

2351. With the notation $\binom{n}{r} \equiv {}^n\mathrm{C}_r$, prove that

$$\binom{n-1}{r} - \binom{n-1}{r-1} \equiv \frac{n-2r}{n}\binom{n}{r}.$$

2352. Show that the derivative of $\sec^2 x$ is the same as the derivative of $\tan^2 x$.

2353. Three forces act on an object, which remains in equilibrium. One force is then removed. Explain carefully why the removal of the largest force will result in the largest magnitude of acceleration.

2354. Sketch $x^2y^4 = 1$.

2355. Variables x and y are related by

$$\int 2y - 1\,dy = \int \frac{1}{x^2}\,dx.$$

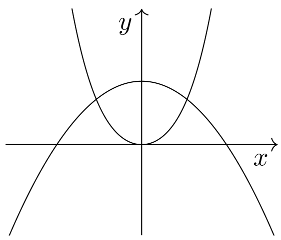
(a) Show that $y^2 - y + \dfrac{1}{x} = c$.

(b) As $x \to \infty$, $y \to 1$. Find c.

(c) Hence, show that $2y = 1 \pm \sqrt{1 - \frac{4}{x}}$.

2356. The function f has instruction $\text{f}(x) = 4x^2 - 8x + 1$, and is defined on the domain $\{x \in \mathbb{R} : 0 \leq x \leq 4\}$. Determine the range of the function.

2357. The diagram shows the curves $y = 10x^4 + 40x^2$ and $y = 70 - 20x^2$.



Determine the area enclosed by the curves.

2358. The acceleration of a particle is $a = -2\cos(\pi t)$. Its average velocity, over $t \in [0, \infty)$, is 5 ms^{-1}. Show that the particle's displacement is given by

$$s = 5t + \frac{2}{\pi^2}\big(\cos(\pi t) - 1\big).$$

2359. The line $y = \frac{1}{2}x + k$ is tangent to the unit circle centred on O. Find the possible values of k.

2360. A set of n six-sided dice are stacked on top of one another on a flat table. Show that the number N of dots visible satisfies $14n + 1 \leq N \leq 14n + 6$.

2361. Show that, for $x \approx \pi$, the linear approximation to the sine function is $\sin x \approx \pi - x$.

2362. A biologist models the size P, in millions, of a bacterial population after t hours by $P = A \times 2^{kt}$, where A and k are constants. After 1 hour, the population is 4.8 million. The population then doubles over the next hour. Find the population predicted by the model after 7 hours.

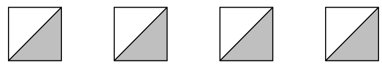
2363. Solve $\displaystyle\sum_{i=0}^{2} \frac{x+i}{x-i} = 0$.

2364. Three lines have equations

$$y = \big(2 \pm \sqrt{3}\big)x, \quad x + y = k.$$

Prove that the pairwise intersections of the lines are equidistant from one another.

2365. The four tiles below are placed together, in random orientations, to form a two-by-two square.



Find the probability that the resulting pattern has rotational symmetry order 4.

2366. The second, third and fourth terms of a geometric sequence are given as $c - 1$, $2c$, $5c + 3$. Find all possible values of the first term.

2367. This question concerns the curve $y = \sec x$, defined over the domain $(-\pi/2, 0)$.

(a) Show that the curve is decreasing.

(b) Show carefully that $\dfrac{dy}{dx} = -y\sqrt{y^2 - 1}$.

2368. Use the substitution $u = 1 + 2x$ to show that

$$\int_0^4 \frac{3x}{\sqrt{1+2x}}\,dx = 10.$$

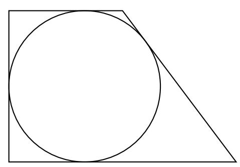
2369. By finding all stationary points of the curve

$$y = \cos^2 x + \cos x$$

in the domain $[0, 2\pi)$, determine the range of

$$\text{f} : x \longmapsto \frac{1}{\cos^2 x + \cos x}.$$

2370. A *tangential polygon* is a convex polygon in which a circle can be inscribed which is tangent to each side of the polygon.



State, with a reason, whether each of the following polygons is necessarily tangential:

(a) a triangle,

(b) a quadrilateral,

(c) a regular n-gon.

2371. Two dice have been rolled, giving scores X and Y. Determine whether the fact "$X = 2Y$" increases, decreases or doesn't change $\mathbb{P}(X + Y) = 7$.

2372. Prove the following identities:

(a) $\tan\left(\frac{\pi}{2} - \theta\right) \equiv \cot\theta$,

(b) $\tan\left(\frac{\pi}{4} + \theta\right) \equiv \cot\left(\frac{\pi}{4} - \theta\right)$.

2373. A function is defined, over a suitable domain, by

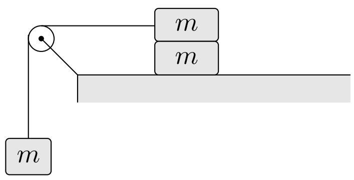
$$\mathrm{f} : x \mapsto \frac{x+a}{x+b}.$$

(a) For $a = 1, b = -1$, show that $\mathrm{f}^{-1} = \mathrm{f}$.

(b) Determine whether there are any other values a and b such that f is self-inverse.

2374. Solve $(x^2+1)^5 + (x^2+1)^4(x-7) = 0$.

2375. A pulley system is set up on a table as depicted. The pulley is light and smooth, and the string is light and inextensible. The coefficient of friction at the upper surface of the lower block is μ_1 and at its lower surface is μ_2.



Find the sets of values of μ_1 and μ_2 for which the system will remain in equilibrium.

2376. Simplify $\dfrac{xz + yz - x^2 - xy}{2xz + 4yz - 2x^2 - 4xy}$.

2377. A sector has arc length $(x+1)$ cm, area x cm^2, and subtended angle $2x$ radians. Determine x.

2378. The quantity $Q = x^2 + y^2$ is to be minimised, under condition C, which is $x - 2y = 10$.

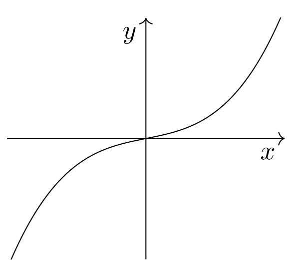
(a) Find the equation of the normal to C that passes through $(0, 0)$.

(b) Find the intersection of this normal with C.

(c) Hence, show that $Q_{\text{min}} = 20$.

2379. Solve the equation $\sqrt{6x-9} + \sqrt{2x-5} = x - 1$.

2380. Show that $y = ax^3 - bx$, for positive constants a, b, could not possibly be the equation of the graph:



2381. The quadratic equation $3x^2 + 2x - 14 = 0$ has two distinct real roots at $x = p, q$. Find a simplified quadratic equation whose roots are at $x = -p, -q$.

2382. A function g has domain $\mathbb{R}$ and range $[0, 1]$. State, with a reason, whether the following, for constants $a, b \in \mathbb{R}$, are necessarily true,

(a) $a\,\mathrm{g}(x) + b \in [b, a+b]$ for all $x \in \mathbb{R}$,

(b) $x \mapsto a\,\mathrm{g}(x) + b$ has range $[b, a+b]$ over $\mathbb{R}$.

2383. If $\dfrac{d}{dx}(x^2 - 2y) = 1$, find $\dfrac{dx}{dy}$ in terms of x.

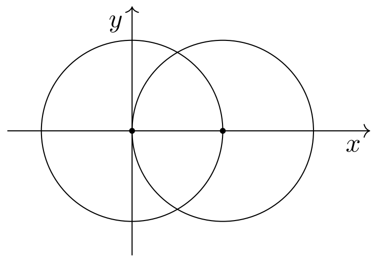
2384. The third time derivative of position is known as the *jerk*. In a collision trial for a new car, a crash test dummy is measured to experience a jerk given by $j = -120$ ms^{-3} for the first 0.5 seconds, having been moving at constant velocity. Determine the total reduction in speed during this period.

2385. In this question, the linear function f, defined by $\mathrm{f}(x) = ax + b$ for $a, b \in \mathbb{R}$, is *self-inverse*, i.e. $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$.

(a) Find all possible values of a and b.

(b) Prove that the graph $y = \mathrm{f}(x)$ has $y = x$ as a line of reflective symmetry.

2386. Two circles C and D are centred on points $(0, 0)$ and $(4, 0)$. Each circle passes through the centre of the other.



(a) Find the coordinates of the intersections.

(b) Show that the area common to both circles is

$$A = \frac{32\pi}{3} - 8\sqrt{3}.$$

2387. A coin is tossed 6 times. Find the probability that no two consecutive tosses yield the same result.

2388. A student has attempted to calculate

$$\int_1^2 (3x-1)^3\,dx.$$

(a) Sketch the area which this represents.

(b) The student's answer is 152.25. Calculate the correct value, and explain the (likely) error the student has made.

2389. A little before 8pm, the angle between the hands of a clock is $\frac{3\pi}{10}$ radians. Find the exact time.

2390. Disprove the following claim regarding the median of a triangle, which is the line joining a vertex to the midpoint of the opposite side: "The median of a triangle bisects the angle."

2391. A *rational function* q has the form

$$\mathrm{q} : x \longmapsto \frac{\mathrm{f}(x)}{\mathrm{g}(x)},$$

where f and g are polynomial. Prove that, if q_1 and q_2 are rational functions, then $x \longmapsto \mathrm{q}_1(x) + \mathrm{q}_2(x)$ is a rational function.

2392. In this question, do not use a calculator.

The equation $3x^3 - 25x^2 + 56x - 16 = 0$ has two distinct real roots.

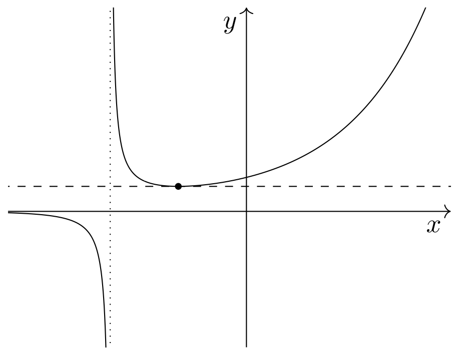
(a) Find k such that the equation can be expressed in the form

$$3x(x-k)^2 - (x-k)^2 = 0.$$

(b) Hence, solve the equation.

2393. The vector $a(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) - \mathbf{k}$ has length 9. Find all possible values of a.

2394. The diagram shows $y = \dfrac{e^x}{x+2}$:



Determine the equation of the

(a) dotted line,

(b) dashed line.

2395. A woman has two sock drawers, one containing 10 pairs of socks bundled as pairs and one containing 10 pairs of socks unbundled as single socks. Each pair is different. She chooses a drawer at random. Then, depending on the drawer, she either picks a bundled pair at random or two single socks at random.

(a) Find the probability that her socks match.

(b) Given that her socks match, determine the probability that she picked a bundled pair.

2396. Write $k^3 + 3k^2 + 4k + 12$ in terms of $(k+1)$.

2397. Sketch the graph $y = x^2(x-a)(x-b)$, where a and b are constants with $0 < a < b$.

2398. True or false?

(a) $|x| \equiv \sqrt{x^2}$,

(b) $|x| \equiv \sqrt[3]{x^3}$,

(c) $|x| \equiv \sqrt[4]{x^4}$.

2399. A curve is given by $y = (2x+5)^4 - (2x+5)^2 + 2$. Determine the coordinates of the local maximum of this curve.

2400. The *Basel problem* was famously solved, in 1735, by Leonhard Euler, showing that

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}.$$

Give the exact values of the following sums:

(a) $\displaystyle\sum_{r=1}^{\infty} \frac{3}{r^2}$,

(b) $\displaystyle\sum_{r=2}^{\infty} \frac{1}{r^2}$,

(c) $\displaystyle\sum_{r=1}^{\infty} \frac{1}{(2r+4)^2}$.

—— End of 24th Hundred ——